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STABILITY OF COUETTE FLOW OF AN IDEAL FLUID WITH FREE BOUNDARIES

V. K. Andreev and A. M. Frank

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The linear and nonlinear stage of development of instability of Couette flow with two free boundaries is studied. It is established that instability occurs only for long waves, and the critical wave number is computed. In the presence of surface-tension forces, instability is preserved only at Weber numbers We $\leqslant 1/3$.

As is known, Couette flow between two solid walls is steady in the case of both an ideal liquid [1] and a viscous liquid [2]. In numerical calculations [3], Frank detected instability of such flow in the presence of two free boundaries.

In the present work, we consider the stability of Couette flow of an ideal liquid layer with free boundaries. Within the framework of the linear theory, it is shown that instability actually occurs only for disturbances with wavenumbers $k < k_* = 1.19968/l$ (l is the half-width of the layer). When capillary forces are taken into account, the instability is preserved for Weber numbers We $\leqslant 1/3$. The nonlinear stage of development of disturbances was studied by the particle method [3]. It turned out that high instability with formation of a vortex chain is observed only for rather long-wave perturbations. Shorter perturbations practically do not grow and do not lead to distortion of the free-boundary shape. The critical number k_*, obtained in the linear theory, is well confirmed in calculations.

1. Linear Problem of Small Perturbations. We consider a layer of thickness $2l$ of an ideal incompressible liquid of constant density ρ. It is assumed that the liquid layer is surrounded by a passive gas, and the straight lines $y = l$ and $y = -l$ are the free boundaries of the layer. It is possible to verify that Couette flow

$$\mathbf{u} = (ay, 0), \quad p = p_0, \quad a, p_0 = \text{const} \tag{1.1}$$

satisfies the Euler equations within the layer and the free-boundary conditions.

Let $U(x,y,t)$, $V(x,y,t)$, and $P(x,y,t)$ be the perturbation of the velocity and pressure of the main flow (1.1). The problem of small perturbations of a liquid flow with a free boundary is generally studied in [4]. Flow (1.1) is described by the system

$$U_t + ayU_x + aV + \frac{1}{\rho}P_x = 0, \quad V_t + ayV_x + \frac{1}{\rho}P_y = 0, \quad U_x + V_y = 0 \tag{1.2}$$

in the layer $-\infty < x < \infty$, $-l < y < l$;

$$P(x, \pm l, t) = 0, \quad R_{1,2t} + ayR_{1,2x} - V(x, \pm l, t) = 0 \tag{1.3}$$

on the free boundaries. In (1.3), the functions $R_1(x,t)$ and $R_2(x,t)$ represent perturbations of the layer boundaries $y = -l$ and $y = l$, respectively.

We seek a solution of problem (1.2) and (1.3) in the form of normal waves:

$$(U, V, P, R_1, R_2) = [U(y), V(y), P(y), R_1, R_2] \exp(-i\omega t + kix). \tag{1.4}$$

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Here k is the wavenumber and $\omega = \omega_r + i\omega_i$ is the complex frequency. The constants R_1 and R_2 on the right side of (1.4) represent the amplitudes of waves propagating along the free boundaries with phase velocity ω_r. Substituting (1.4) into (1.2) and (1.3), we obtain

$$i(ayk-\omega)U + aV + \frac{ki}{\rho}P = 0,\quad i(ayk-\omega)V + \frac{1}{\rho}P_y = 0,\quad ikU + V_y = 0 \quad \text{for} \quad -l < y < l; \tag{1.5}$$

$$P(\pm l) = 0, \qquad i(ayk-\omega)R_{1,2} - V(\pm l) = 0. \tag{1.6}$$

In studies of the stability of plane–parallel flows with solid walls, perturbations of the pressure P and the longitudinal velocity U are usually eliminated, and a spectral problem for the component V — the Rayleigh problem — is obtained. In our case, by virtue of the specificity of boundary conditions (1.6), it is more convenient to eliminate the functions U and V:

$$V = \frac{i}{\rho(ay-\omega/k)k}P', \qquad U = \frac{i}{k}V' \tag{1.7}$$

(the prime denotes differentiation with respect to y). Substituting (1.7) into the first equation of (1.5), we obtain the equation for the perturbed pressure

$$P'' - \frac{2a}{ay-\omega/k}P' - k^2P = 0 \tag{1.8}$$

with the boundary conditions

$$P(-l) = P(l) = 0. \tag{1.9}$$

For a known function $P(y)$, the free-boundary perturbations are obtained from the second group of Eqs. (1.6).

We introduce the following dimensionless variables and parameters:

$$z = \frac{y}{l} - \frac{\omega}{kla}, \qquad n = kl, \qquad q = \frac{\omega}{a}. \tag{1.10}$$

In this case, Eq. (1.8) takes the form $zP'' - 2P' - n^2zP = 0$. It has the general solution

$$P = \frac{z}{\sqrt{in}}\Big\{C_1\Big[\cos(inz) - \frac{1}{inz}\sin(inz)\Big] + C_2\Big[-\sin(inz) - \frac{1}{inz}\cos(inz)\Big]\Big\} \tag{1.11}$$

where C_1 and C_2 are arbitrary constants.

After simple but rather long calculations using boundary conditions (1.9), we obtain the following secular equation for q^2:

$$q^2 = n^2 + 1 - 2n\coth(2n). \tag{1.12}$$

It is easy to verify the following properties of the function $q^2(n)$:

1) $\lim_{n\to 0} q^2(n) = 0, \qquad \lim_{n\to 0}\frac{dq^2(n)}{dn} = 0, \qquad \lim_{n\to 0}\frac{d^2q^2(n)}{dn^2} = -\frac{2}{3},$

2) $q^2(n) \simeq -\frac{1}{3}n^2$ for $n \to 0$,

3) $q^2(n) \simeq n^2$ for $n \to \infty$.

A plot of the function $q^2(n)$ is shown in Fig. 1 (curve 1). Since $q^2 = (1 - n\tanh n)(1 - n\coth n)$, then n_* is actually a solution of the equation $\tanh n = 1/n$ and $n_0 \approx 0.8$ is the minimum point of the function $q^2(n)$, i.e., $q^2(n) \approx -9.6\cdot 10^{-2}$.

Therefore, Couette flow (1.1) is always unstable for $0 < n < n_* \approx 1.19968$ and steady for $n \geqslant n_*$. Reverting to (1.10), it is possible to draw the following conclusions:

1. If the perturbation wavelength $\lambda > 2\pi l/n_* \approx 5.23738l$, flow (1.1) in the layer is unstable and the growth coefficient ω_i decreases together with λ when $\lambda \leqslant \lambda_0 = 2\pi l/n_0 \approx 7.85398l$ and $\omega_{i\,\max} = a|q(n_0)| \approx 0.30984a$. For $\lambda > \lambda_0$, ω_i increases.

2. If the perturbation wavelength $\lambda \leqslant 2\pi l/n_*$, the flow (1.1) is steady.

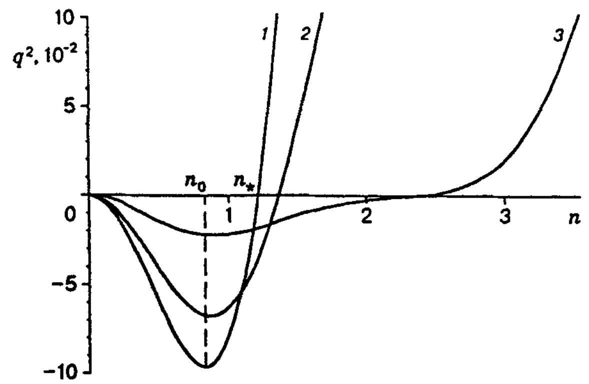

Fig. 1

2. Effect of Surface-Tension Forces. When surface forces are taken into account, the pressure on the perturbed free boundaries is proportional to their mean curvatures. Therefore, instead of the first group of boundary condition (1.3), we have

$$P - \sigma R_{1xx} = 0 \quad \text{for} \quad y = -l, \qquad P + \sigma R_{2xx} = 0 \quad \text{for} \quad y = l \tag{2.1}$$

($\sigma > 0$ is the surface-tension coefficient). Using the second condition of (1.6), the first equation of (1.7) and expression (1.11), instead of (1.9) we obtain the following conditions for the amplitudes of pressure perturbations:

$$P(z_2) - \frac{\text{We}}{z_2^2}\frac{dP}{dz}(z_2) = 0, \qquad P(z_1) + \frac{\text{We}}{z_1^2}\frac{dP}{dz}(z_1) = 0. \tag{2.2}$$

Here $z_1 = -1 - q/n$, $z_2 = 1 - q/n$, $q = \omega/a$, $n = kl$, and $\text{We} = \sigma/\rho l^3 a^2$ (We is a Weber number).

After substitution of the expression for $P(z)$ from (1.11) into boundary conditions (2.2) and some transformations we obtain the secular equation

$$q^2 = n^2 + \frac{1}{2}\Bigg[2n^3 \coth 2n\text{We} - 2n\coth 2n + 1$$

$$\pm\sqrt{\frac{4n^6}{\sinh^2 2n}\text{We}^2 + 4n^3(\coth 2n - 2n)(1 - 2n\coth 2n)\text{We} + (2n\coth 2n - 1)^2}\;\Bigg]. \tag{2.3}$$

For short waves ($n \to \infty$), $q \simeq \pm\sqrt{n^3\text{We}}$ and stability takes place. For long waves ($n \to 0$), from (2.3) we have $q^2 \simeq n^2(\text{We} - 1/3)$ if $\text{We} \geqslant 4/3$ and

$$q^2 \simeq n^2\Big\{\text{We} - \frac{1}{3} + 2\Big[\frac{\text{We}}{3} - \frac{4}{45} - \frac{2}{|\text{We}/4 - 1/3|}\Big(-\frac{\text{We}^2}{48} + \frac{11}{90}\text{We} - \frac{2}{135}\Big)\Big]n^2\Big\}$$

if $0 < \text{We} < 4/3$. Therefore, for $\text{We} \leqslant 1/3$, there is long-wave instability of Couette flow (1.1). Moreover, in this case there is always a value of n_* such that $q^2(n) < 0$ for $n < n_*$. For example, for $\text{We} = 0.1$ (curve *2* in Fig. 1), we have $n_* \approx 1.305$, $n_0 \approx 0.84$, and $q^2(n_0) \approx -6.67 \cdot 10^{-2}$ and for $\text{We} = 0.25$ (curve 3), we have $n_* \approx 2.3675$, $n_0 \approx 0.88$, and $q^2(n_0) \approx -2.17 \cdot 10^{-2}$. If $\text{We} > 1/3$, the function $q^2(n) \geqslant 0$ for all wavenumbers.

Remark 1. Let $y = 0$ be an impermeable solid wall. Then, instead of (2.2) we have the conditions

$$P(z_2) - \frac{\text{We}}{z_2^2}\frac{dP}{dz}(z_2) = 0, \qquad \frac{dP}{dz}\Big(-\frac{q}{n}\Big) = 0$$

can show that the perturbation frequency is always real:

$$\frac{\omega}{a} = q = \frac{2n - \tanh n \pm \sqrt{\tanh^2 n + 4n^3 \tanh n\,\text{We}}}{2}. \tag{2.4}$$

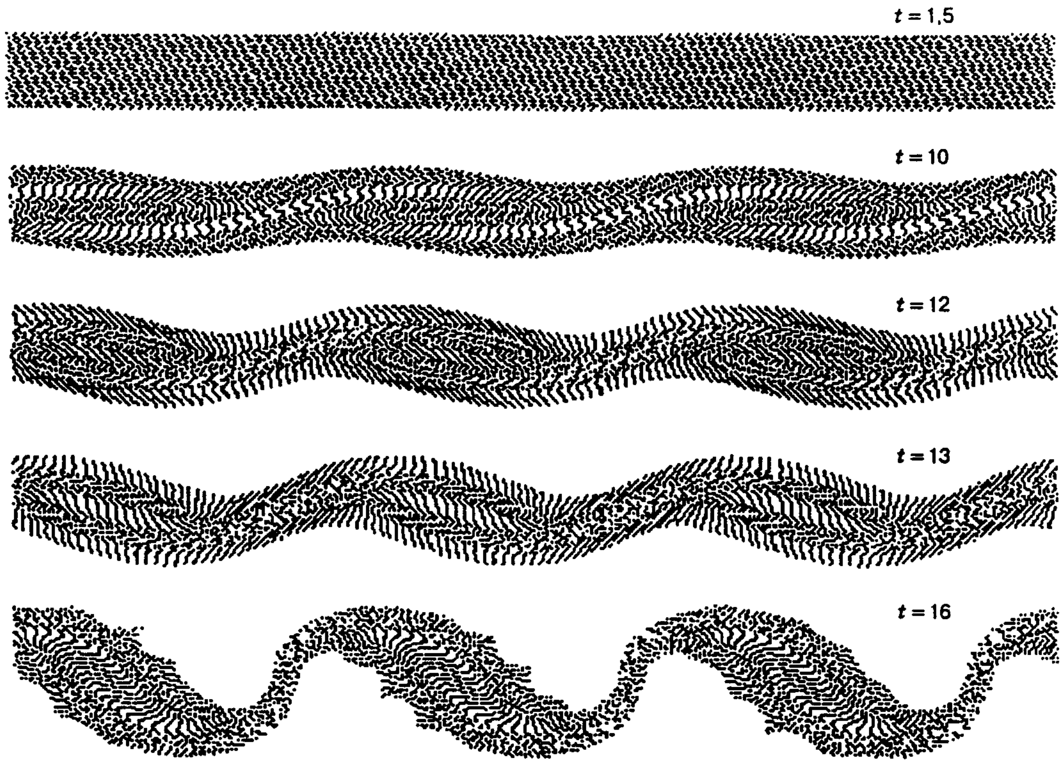

Fig. 2

Thus, the presence of only two free boundaries can lead to instability of Couette flow (1.1).

Remark 2. Let $\bar{V}(k, y, t)$ be a Fourier transform of the function $V(x, y, t)$ with respect to the variable x. Then, the solution of the initial boundary-value problem that corresponds to (1.2) and (1.3) has the form

$$\bar{V} = -\frac{1}{k}\int_y^l \exp(-ika\xi t) f(\xi) \sinh k(y-\xi)\, d\xi + [c_1(t)\, e^{iqat} + c_2(t)\, e^{-iqat}] \sinh ky + [d_1(t)\, e^{iqat} + d_2(t)\, e^{-iqat}] \cosh ky,$$

where $f(y) = \bar{V}_{0yy} - k^2\bar{V}_0$ [$\bar{V}_0(y)$ is a Fourier transform of the initial value $V_0(x, y)$]; $c_1(t)$, $c_2(t)$, $d_1(t)$, and $d_2(t)$ are limited functions as $t \to \infty$; the parameter q is determined by one of values (1.12), (2.3), or (2.4). Reverting to the second boundary condition of (1.3), we obtain growth or damping of initial perturbations, as in the method of elementary wave solutions.

3. Nonlinear Stage. The nonlinear development of perturbations was studied by the particle method for an incompressible liquid [5]. The method is based on simulation of liquid flows by means of a great number of material particles. The particles are free, i.e., they are not attached to a grid and, at the same time, they always move at a solenoidal speed, which ensures incompressibility of the flow. Equations of motion that are discrete in t are derived from the Gauss variational principle, and, hence, the method is completely conservative. One step in time consists, in essence, of two fractional steps, of which the first is the free motion of the particles under the action of external forces, and the second is the projection in L_2 of the resulting discrete velocity field onto a certain finite-dimensional space H of smooth solenoidal functions. In calculations, this space is specified by means of a basis which is usually constructed using B splines. A detailed description of the method and examples of problems solved, including test problems can be found in [3, 6, 7]. We point out some features typical of the problem considered.

The unperturbed liquid layer with two flat free boundaries $y = \pm l$ has thickness $2l$ and the linear velocity field (1.1) with $a = 1$, $u = y$, and $v = 0$. At the initial instant, particles in the layer are uniformly distributed along every coordinate. As the basis of space H we use finite solenoidal functions obtained by

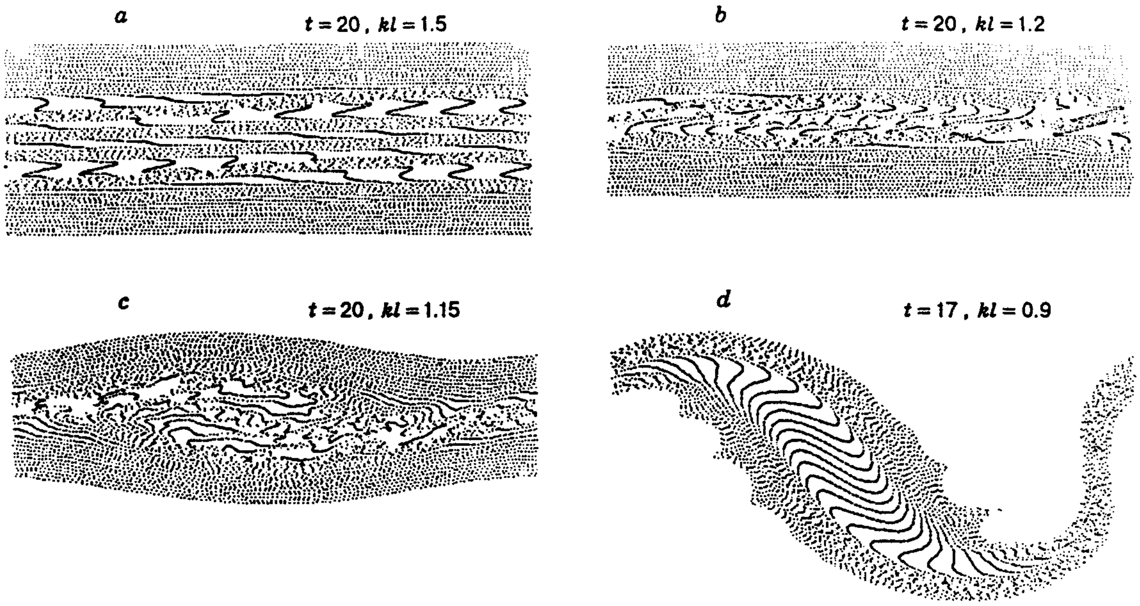

Fig. 3

applying the rot operator to two-dimensional quadratic B splines. The splines are constructed on a uniform rectangular grid with steps h_1 and h_2. The length of the computational domain is equal to 2π. On its lateral boundaries, we impose periodicity conditions, which are natural for the problem considered. In the method described, they are realized are realized as follows. A particle that has reached the right boundary of the domain enters it at the left at once, as if the lateral boundaries of the computational domain are joined together. The basis functions that lie near the right boundary participate in the formation of the velocity field near the left boundary and vice versa.

In the absence of initial perturbations, such flow with a linear velocity distribution is a practically exact solution for the given method (with accuracy up to convergence of iterations in the solution of the linear system in the projection problem). Therefore, the particle distribution remains uniform during the calculation, without bending of the layers sliding relative to each other, at least up to $t = 20$, up to which the calculations of perturbed flows were performed.

In simulating instability at the initial instant, we impose the following perturbations on the linear velocity field:

$$\delta u = 0, \qquad \delta v = 0.01 l \sin(kx). \tag{3.1}$$

Figure 2 shows an example of calculation of the development in time of rather long unstable disturbances with a dimensionless wavenumber $kl = 0.9$. The calculation parameters are as follows: layer thickness $2l = 0.6$, a 60×25 grid with steps $h_1 = 0.105$ and $h_2 = 0.1$, number of particles 5880, step in time $\tau = 0.1$, time of computing one step on AT 486DX2 about 8 sec. Initially, at $t = 1.5$, the free-boundary perturbations are slight. On the contrary, the perturbations of the flow field inside the layer are clearly seen because of small displacement of particles in the originally regular structure. Further, at $t > 10$, the perturbations increase so that the flow is rearranged, forming a chain of vortices that are very similar to Kelvin–Helmholtz vortices, which arise on the interface in a two-layer liquid [8, p. 87]. Rotation of these vortices on the boundary leads to occurrence of secondary Rayleigh–Taylor instability, which is clearly seen at $t = 16$ (see also Fig. 3d).

Figure 3 shows results of calculation of perturbation propagation for different values of the dimensionless wavenumber kl at the same time $t = 20$, except for case 3d, where perturbations grow too fast. The perturbations have the form (3.1) with a wavenumber $k = 1$, i.e., the wavelength is equal to 2π, and the amplitude amounts to 1% of the maximum velocity in the layer $U = l$. The value of the wavenumber was determined by varying the layer thickness. The characteristic time scale $T = l/U$ remained unchanged. It is evident that for $kl \geqslant 1.2$, the perturbations do not lead to distortion of the shape of the layer. The amplitude of velocity-field perturbations in case 3a remains constant, about 1%, and in case 3b, it increases to 2%. The strong perturbations of the internal structure of the layer, which are clearly seen in the figures, practically in no way characterize the instantaneous velocity field. They are due to the slight bending of the liquid layers under the action of initial small perturbations and their subsequent deformation (of the type of breaking) under the action of shear flow. For $kl < 1.2$, the perturbations grow fast, forming a periodic vortex structure. Thus, the critical value of the wavenumber obtained from the linear theory $kl = 1.19968$ agrees very well with the numerical results.

In conclusion, some words should be said about the control of the integral characteristics of the flow. Although, formally, the method is completely conservative, in real calculations, the system of simple equations is solved by iterative methods, and this introduces inaccuracy into the conservation laws. Direct verification showed that in the calculations illustrated in Fig. 3, the deviation of the horizontal momentum of the flow from the doubled momentum of half of the layer (since the total momentum is equal to zero) was not more than 0.1%, and the relative deviation of the kinetic energy was less than 0.03%. We also monitored conservation of the total circulation, which varied within 0.01–4%, depending on the variant. The greatest error arose here in variant 3d where sharp peaks occurred at the free boundary.

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